

Characterization of exponential distribution through equidistribution conditions for consecutive maxima

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Abstract

A characterization of the exponential distribution based on equidistribution conditions for maxima of random samples with consecutive sizes $n - 1$ and n for an arbitrary and fixed $n \geq 3$ is proved. This solves an open problem stated recently in Arnold and Villasenor [3].

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1 Introduction

Characterizations of the exponential distribution are abundant. Comprehensive surveys can be found in Ahsanullah and Hamedani [1], Arnold and Huang [2], and Johnson, Kotz, and Balakrishnan [5]. Recently, Arnold and Villasenor [3] obtained a series of characterizations based on random sample of size two. In particular, they proved under some additional regularity conditions, that for a random sample X_1 and X_2 of a continuous parent X

$$X_1 + \frac{1}{2}X_2 \stackrel{d}{=} \max\{X_1, X_2\}, \quad (1)$$

where $\stackrel{d}{=}$ denotes equality in distribution is a necessary and sufficient condition for X to be exponentially distributed. Arnold and Villasenor [3] also conjectured that for a random sample X_1, X_2 and X_3 of a continuous parent X if

$$\max\{X_1, X_2\} + \frac{1}{3}X_3 \stackrel{d}{=} \max\{X_1, X_2, X_3\}, \quad (2)$$

then X is exponentially distributed. We confirm and extend this last conjecture for a sample of any fixed size $n \geq 2$. Note that in Yanev and Chakraborty (2013) the case (2) was considered.

Let X_1, X_2, \dots, X_n , $n \geq 2$ be a random sample from an exponentially

distributed parent X . It is known (e.g., Nevzorov (2001), p.11) that

$$\max\{X_1, X_2, \dots, X_{n-1}\} + \frac{1}{n}X_n \stackrel{d}{=} \max\{X_1, X_2, \dots, X_n\}. \quad (3)$$

We write $X \sim \exp(\lambda)$ if the probability density function (pdf) of X equals $f_X(x) = \lambda e^{-\lambda x} I(x > 0)$. Our goal is to prove that (3), under some regularity assumptions on the cumulative distribution function (cdf) F of X , is also a sufficient condition for X to be exponential.

Theorem Let X be a non-negative continuous random variable with pdf f . If f has derivatives of all orders in a neighborhood of zero and (3) holds true, then $X \sim \exp(\lambda)$ with some $\lambda > 0$.

Wesołowski and Ahsanullah (2004) and more recently Castaño-Martínez et al. (2012) proved characterizations of probability distributions in the context of random translations. The theorem above can be deduced from their results (see Corollary 1 in Wesołowski and Ahsanullah (2004) and Corollary 3 in Castaño-Martínez et al. (2012)). However, the proof [presented here is different from those in the above papers, where the authors make use of certain recurrences between order statistics and integral equations. The direct technique of our proof may also be used in obtaining some more general results, possibilities which will be explored in the future.

2 Preliminaries

Define for all non-negative integers n , i , and any real number x

$$H_{n,i}(x) := \sum_{j=0}^n (-1)^j \binom{n}{j} (x-j)^i.$$

It is known, (e.g., Ruiz 1996) that for all integers $n \geq 0$ and all real x

$$H_{n,i}(x) = \begin{cases} n! & \text{if } i = n; \\ 0 & \text{if } 0 \leq i \leq n-1. \end{cases} \quad (4)$$

Define $G_m(x) := F^m(x)f(x)$ for $m \geq 1$ and denote by $g^{(i)}(x)$ for $i \geq 1$ the i th derivative of a function $g(x)$; $g^{(0)}(x) := g(x)$.

Lemma 1 If for $0 \leq r \leq m-1$

$$f^{(r)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{r-1} f'(0), \quad (5)$$

then for $0 \leq i \leq 2m$

$$G_m^{(i)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{i-m} f^{m+1}(0) H_{m,i}(m+1). \quad (6)$$

Proof. Case $0 \leq i \leq m-1$. In this case (4) implies $H_{m,i}(m+1) = 0$. On the other hand, in the left-hand side of (6), we have $G_m^{(i)}(0) = 0$ because all the terms in the expansion of $G_m^{(i)}(0)$ has a factor $F(0) = 0$.

Case $\mathbf{i} = \mathbf{m}$. From (4) it follows that (6) is equivalent to

$$G_m^{(m)}(0) = m!f^{m+1}(0). \quad (7)$$

Assuming that (7) is true for $m = k$, we will prove it for $m = k + 1$. Since $G_{k+1}(x) = F(x)G_k(x)$, we have

$$\begin{aligned} G_{k+1}^{(k+1)}(0) &= \sum_{j=0}^{k+1} \binom{k+1}{j} F^{(j)}(0) G_k^{(k+1-j)}(0) \\ &= F(0) G_k^{(k+1)}(0) + (k+1) F^{(1)}(0) G_k^{(k)}(0) \\ &= (k+1) f(0) k! f^{k+1}(0) \\ &= (k+1)! f^{k+2}(0), \end{aligned}$$

which completes the proof in this case.

Case $\mathbf{m} < \mathbf{i} \leq 2\mathbf{m}$. Suppose we have proved (6) for $m = 1, 2, \dots, k$. We want to prove it for $m = k + 1$. We have

$$G_{k+1}^{(i)}(0) = \sum_{j=0}^i \binom{i}{j} F^{(j)}(0) G_k^{(i-j)}(0)$$

Since $G_k^{(r)}(0) = 0$ for $0 \leq r \leq k - 1$, making use of (5) and the induction assumption, we obtain

$$G_{k+1}^{(i)}(0) = \sum_{j=1}^k \binom{i}{j} f^{(j-1)}(0) G_k^{(i-j)}(0) + \sum_{j=k+1}^i \binom{i}{j} f^{(j-1)}(0) G_k^{(i-j)}(0) \quad (8)$$

$$\begin{aligned}
&= \sum_{j=1}^k \binom{i}{j} \left[\frac{f'(0)}{f(0)} \right]^{j-2} f'(0) \left[\frac{f'(0)}{f(0)} \right]^{i-j-k} f^{k+1}(0) H_{k,i-j}(k+1) \\
&= \left[\frac{f'(0)}{f(0)} \right]^{i-k-1} f^{k+2}(0) \sum_{j=1}^i \binom{i}{j} H_{k,i-j}(k+1),
\end{aligned}$$

where in the last equality we used that by (4) $H_{k,i-j}(k+1) = 0$ for $j = i+1, \dots, k$. Further, we have

$$\begin{aligned}
\sum_{j=1}^i \binom{i}{j} H_{k,i-j}(k+1) &= \sum_{r=0}^k (-1)^r \binom{k}{r} \sum_{j=1}^i \binom{i}{j} (k+1-r)^{i-j} \\
&= \sum_{r=0}^k (-1)^r \binom{k}{r} [(k+2-r)^i - (k+1-r)^i] \\
&= (k+2)^i - \left[(k+1)^i + \binom{k}{1} (k+1)^i \right] + \left[\binom{k}{1} k^i + \binom{k}{2} k^i \right] \\
&\quad + \dots + (-1)^k \left[\binom{k}{k-1} 2^i + 2^i \right] + (-1)^{k+1} \\
&= (k+1)^i - \binom{k+1}{1} (k+1)^i + \dots + (-1)^k \binom{k+1}{k} 2^i + (-1)^{k+1} \\
&= \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (k+2-j)^i \\
&= H_{k+1,i}(k+2).
\end{aligned}$$

The lemma's claim follows by induction, taking into account (8) and (9).

The identity below may be a independent interest.

Lemma 2 For any integer $m \geq 0$ and $k \geq 0$

$$\sum_{j=0}^m (k+2)^{m-j} H_{k,j}(k+1) = \sum_{j=0}^m \binom{m+1}{j+1} H_{k,j}(k+1). \quad (9)$$

Proof. The left-hand side of (9) equals

$$\begin{aligned}
& \sum_{j=0}^m (k+2)^{m-j} \sum_{i=0}^k (-1)^i \binom{k}{i} (k+1-i)^j \\
&= \sum_{i=0}^k (-1)^i \binom{k}{i} (k+2)^m \sum_{j=0}^m \left(\frac{k+1-i}{k+2} \right)^j \\
&= \sum_{i=0}^k (-1)^i \binom{k}{i} \frac{1}{i+1} [(k+2)^{m+1} - (k+1-i)^{m+1}] \\
&= \sum_{i=0}^k (-1)^i \binom{k+1}{i+1} \frac{1}{k+1} [(k+2)^{m+1} - (k+1-i)^{m+1}] \\
&= -\frac{(k+2)^{m+1}}{k+1} \sum_{r=1}^{k+1} (-1)^r \binom{k+1}{r} + \frac{1}{k+1} \sum_{r=1}^{k+1} (-1)^r \binom{k+1}{r} (k+2-r)^{m+1} \\
&= -\frac{(k+2)^{m+1}}{k+1} \left[\sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} - 1 \right] \\
&\quad + \frac{1}{k+1} \left[\sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (k+2-r)^{m+1} - (k+2)^{m+1} \right] \\
&= \frac{1}{k+1} \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (k+2-r)^{m+1}.
\end{aligned} \tag{10}$$

For the right-hand side of (9) we obtain

$$\begin{aligned}
& \sum_{j=0}^m \binom{m+1}{j+1} \sum_{i=0}^k (-1)^i \binom{k}{i} (k+1-i)^j \\
&= \sum_{i=0}^k (-1)^i \binom{k}{i} \sum_{j=0}^m \binom{m+1}{j+1} (k+1-i)^j \\
&= \sum_{i=0}^k (-1)^i \binom{k}{i} \frac{1}{k+1-i} \sum_{j=0}^m \binom{m+1}{j+1} (k+1-i)^{j+1} \\
&= \frac{1}{k+1} \sum_{i=0}^k (-1)^i \binom{k+1}{i} \sum_{r=1}^{m+1} \binom{m+1}{r} (k+1-i)^r
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{k+1} \sum_{i=0}^k (-1)^i \binom{k+1}{i} \left[\sum_{r=0}^{m+1} \binom{m+1}{r} (k+1-i)^r - 1 \right] \\
&= \frac{1}{k+1} \sum_{i=0}^k (-1)^i \binom{k+1}{i} (k+2-i)^{m+1} - \frac{1}{k+1} \left[\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} - (-1)^{k+1} \right] \\
&= \frac{1}{k+1} \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (k+2-r)^{m+1},
\end{aligned}$$

which equals (10). The proof of the lemma is complete.

Next lemma (see also Arnold and Villaseñor [3]) will play a crucial role in the proof of the theorem.

Lemma 3 If $F(0) = 0$, the pdf f has derivatives of all order in a neighborhood of 0, and

$$f^{(k)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{k-1} f'(0), \quad k = 1, 2, \dots, \quad (11)$$

then $X \sim \exp\{\lambda\}$ for some $\lambda > 0$.

Proof. For the Maclaurin series of $f(x)$, we have for $x > 0$

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
&= f(0) + \sum_{k=1}^{\infty} \left[\frac{f'(0)}{f(0)} \right]^{k-1} f'(0) \frac{x^k}{k!} \\
&= f(0) \exp \left\{ \frac{f'(0)}{f(0)} x \right\}.
\end{aligned} \quad (12)$$

Since $f(x)$ is a pdf, we have $f'(0)/f(0) < 0$. Denoting $\lambda = -f'(0)/f(0) > 0$

and setting the integral of (12) from 0 to ∞ to be 1, we obtain $\lambda = f(0)$.

Therefore, $f(x) = \lambda e^{-\lambda x} I(x > 0)$, i.e., $X \sim \exp\{\lambda\}$.

3 Proof of the theorem

Equation (3) can be written as

$$\int_0^x f_{X_n/n}(y) f_{\max\{X_1, \dots, X_{n-1}\}}(x-y) dy = n(n-1)f(x) \int_0^x G_{n-2}(y) dy.$$

This is equivalent to

$$\int_0^x n f(ny)(n-1)F^{n-2}(x-y)f(x-y) dy = n(n-1)f(x) \int_0^x G_{n-2}(y) dy,$$

which simplifies to

$$\int_0^x f(ny)G_{n-2}(x-y) dy = f(x) \int_0^x G_{n-2}(y) dy. \quad (13)$$

Differentiating the left-hand side of (13) $2n-2$ times, we obtain

$$\begin{aligned} & \frac{d^{2n-2}}{dx^{2n-2}} \int_0^x f(ny)G_{n-2}(x-y) dy \\ &= \sum_{i=0}^{2n-3} n^{2n-3-i} f^{(2n-3-i)}(x) G_{n-2}^{(i)}(0) + \int_0^x f(ny)G_{n-2}^{(2n-2)}(x-y) dy. \end{aligned} \quad (14)$$

Applying to the right-hand side the Leibnitz product rule of differentiation, we have

$$\begin{aligned} & \frac{d^{2n-2}}{dx^{2n-2}} \left[f(x) \int_0^x G_{n-2}(y) dy \right] \\ &= \sum_{i=0}^{2n-3} \binom{2n-2}{i+1} f^{(2n-3-i)}(x) G_{n-2}^{(i)}(x) + f^{(2n-2)}(x) \int_0^x G_{n-2}(y) dy \end{aligned} \quad (15)$$

Setting $x = 0$ in (14) and (15) and taking into account that $G_{n-2}^{(i)}(0) = 0$ for $0 \leq i \leq n-3$, we obtain that (13) is equivalent to

$$\sum_{i=n-2}^{2n-4} n^{2n-3-i} f^{(2n-3-i)}(0) G_{n-2}^{(i)}(0) = \sum_{i=n-2}^{2n-4} \binom{2n-2}{i+1} f^{(2n-3-i)}(0) G_{n-2}^{(i)}(0).$$

For $i = n-2$, we have $f^{(n-1)}(0) G_{n-2}^{(n-2)}(0) = f^{(n-1)}(0) f^{n-1}(0) (n-2)!$. Thus, the equation above can be written as

$$\begin{aligned} & \left[n^{n-1} - \binom{2n-2}{n-1} \right] f^{(n-1)}(0) f^{n-1}(0) (n-2)! \\ &= \sum_{i=n-1}^{2n-4} \left[\binom{2n-2}{i+1} - n^{2n-3-i} \right] f^{(2n-3-i)}(0) G_{n-2}^{(i)}(0). \end{aligned} \quad (16)$$

In view of Lemma 3, to complete the proof it suffices to show

$$f^{(r)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{r-1} f'(0), \quad r = 1, 2, \dots \quad (17)$$

Assume (17) for all $1 \leq r \leq n-2$. We shall prove it for $r = n-1$, i.e.,

$$f^{(n-1)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{n-2} f'(0), \quad r = 1, 2, \dots \quad (18)$$

It follows from Lemma 1 with $m = n-2$ that for $n-1 \leq i \leq 2n-4$

$$f^{(2n-3-i)}(0) G_{n-2}^{(i)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{i-n+2} f^{n-1}(0) H_{n-2,i}(n-1). \quad (19)$$

Substituting (19) in the right-hand side of (16) we obtain

$$\begin{aligned} & \left[n^{n-1} - \binom{2n-2}{n-1} \right] f^{(n-1)}(0) (n-2)! \\ &= \left[\frac{f'(0)}{f(0)} \right]^{n-2} f'(0) \sum_{i=n-1}^{2n-4} \left[\binom{2n-2}{i+1} - n^{2n-3-i} \right] H_{n-2,i}(n-1). \end{aligned}$$

To establish (19) we need to prove

$$\left[n^{n-1} - \binom{2n-2}{n-1} \right] = \sum_{i=n-1}^{2n-4} \left[\binom{2n-2}{i+1} - n^{2n-3-i} \right] H_{n-2,i}(n-1)$$

or equivalently

$$\sum_{i=n-2}^{2n-4} n^{2n-3-i} H_{n-2,i}(n-1) = \sum_{i=n-2}^{2n-4} \binom{2n-2}{i+1} H_{n-2,i}(n-1). \quad (20)$$

Since (4) implies $H_{n-2,i}(n-1) = 0$ for $0 \leq i \leq n-3$ and for $i = 2n-3$ we have $n^{2n-3-i} = \binom{2n-2}{i+1} = 1$, we obtain that (20) is equivalent to

$$\sum_{i=0}^{2n-3} n^{2n-3-i} H_{n-2,i}(n-1) = \sum_{i=0}^{2n-3} \binom{2n-2}{i+1} H_{n-2,i}(n-1),$$

which follows from Lemma 3 with $m = 2n-3$. This complete the induction argument and thus proves (17). Referring to (17) and Lemma 2 we complete the proof of the theorem.

References

- [1] Ahsanullah, M. and Hamedani, G.G. (2010) Exponential Distribution: Theory and Methods, NOVA Science, New York.
- [2] Arnold, B.C. and Huang, J.S. (1995) Chapter12: Characterizations, In: Balakrishnan, N. and Basu, A.P. (eds.), The Exponential Distribution: Theory, Methods and Applications, Gordon and Breach, Amsterdam, pp. 185-203.
- [3] Arnold, B.C. and Villasenor, J.A. (2013) Exponential characterizations motivated by the structure of order statistics in sample of size two. Statistics and Probability Letters, Vol. 83, 2:596-601.

- [4] Castaño-Martínez, A., López-Blázquez, F., Salamanea-Miño, B. (2012) Random translations, contractions and dilations of order statistics and records. *Statistics*, Vol. 46, 1:57-67.
- [5] Johnson, N.L., Kotz, S., and Balakrishnan, N. (1994) *Continuous Univariate Distributions*, Vol. 1, 2nd Edn., Wiley, New York.
- [6] Nevzorov, V. (2001) *Records: Mathematical Theory*, American Mathematical Society, Providence, Rhode Island.
- [7] Ruiz, SM. (1996) An algebraic identity leading to Wilson's theorem. *The Mathematical Gazette*, Vol. 80, 489:579-582.
- [8] Wesołowski, J. and Ahsanullah, M. (2004) Switching order statistics through random power contractions. *Aust. N.Z.J. Stat.* 46, 2:297-303.
- [9] Yanev, G., Chakraborty, S. (2013) Characterizations of exponential distribution based on sample of size three. *Pliska Stud. Math. Bulgarica*, 23, in print.